

TWISTED EXPONENTIAL SUMS OF POLYNOMIALS IN ONE VARIABLE

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ABSTRACT. The twisted T -adic exponential sum associated to a polynomial in one variable is studied. An explicit arithmetic polygon in terms of the highest two exponents of the polynomial is proved to be a lower bound of the Newton polygon of the C -function of the twisted T -adic exponential sum. This bound gives lower bounds for the Newton polygon of the L -function of twisted p -power order exponential sums.

1. INTRODUCTION

Let p be a prime number, q a power of p , and \mathbb{F}_q the finite field with q elements. Let W be the Witt ring scheme, $\mathbb{Z}_q = W(\mathbb{F}_q)$, and $\mathbb{Q}_q = \mathbb{Z}_q[\frac{1}{p}]$. Let μ_{q-1} be the group of $(q-1)$ -th roots of unity in \mathbb{Z}_q , $\omega : x \mapsto \hat{x}$ the Teichmüller lifting from \mathbb{F}_q to μ_{q-1} , and $\chi = \omega^{-u}$ with $u \in \mathbb{Z}^n/(q-1)$ a character of $(\mathbb{F}_q^\times)^n$ into μ_{q-1} .

Let $\Delta \supsetneq \{0\}$ be an integral convex polytope in \mathbb{R}^n , and I the set of vertices of Δ different from the origin. Let

$$f(x) = \sum_{u \in \Delta} (a_u x^u, 0, 0, \dots) \in W(\mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]) \text{ with } \prod_{u \in I} a_u \neq 0,$$

where $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ if $u = (u_1, u_2, \dots, u_n) \in \mathbb{Z}^n$.

Definition 1.1. For any positive integer l , the sum

$$S_{f,\chi}(l, T) = \sum_{x \in (\mathbb{F}_q^\times)^n} \chi(\text{Norm}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(x)) (1+T)^{\text{Tr}_{\mathbb{Q}_{q^l}/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_q[[T]]$$

is called a twisted T -adic exponential sum of $f(x)$. And the function

$$L_{f,\chi}(s, T) = \exp\left(\sum_{l=1}^{\infty} S_{f,\chi}(l, T) \frac{s^l}{l}\right)$$

is called a L -function of twisted T -adic exponential sums.

We have

$$L_{f,\chi}(s, T) = \prod_{x \in |\mathbb{G}_m^n \otimes \mathbb{F}_q|} \frac{1}{1 - \chi(\text{Norm}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(x)) (1+T)^{\text{Fr}_x s^m}},$$

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where \mathbb{G}_m is the multiplicative group $xy = 1$, $m = \deg(x)$ and $\text{Fr}_x = \text{Tr}_{\mathbb{Q}_{q^m}/\mathbb{Q}_p}(f(x))$.

That Euler product formula gives

$$L_{f,\chi}(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]].$$

The theory of T -adic exponential sums without twists was developed by Liu-Wan [LWn], and the theory of twisted T -adic exponential sums was developed by Liu [Liu2].

Let $m \geq 1$, ζ_{p^m} a primitive p^m -th root of unity, and $\pi_m = \zeta_{p^m} - 1$. Then the specialization $L_{f,\chi}(s, \pi_m)$ is the L -function of twisted p -power order exponential sums $S_{f,\chi}(l, \pi_m)$. These sums were studied by Liu [Liu], with the $m = 1$ case studied by Adolphson-Sperber [AS, AS2], and if χ is trivial, they were studied by Liu-Wei [LW].

Define

$$C_{f,\chi}(s, T) = \exp\left(\sum_{l=1}^{\infty} -(q^l - 1)^{-n} S_{f,\chi}(l, T) \frac{s^l}{l}\right).$$

Call it a C -function of twisted T -adic exponential sums.

We have

$$L_{f,\chi}(s, T) = \prod_{i=0}^n C_{f,\chi}(q^i s, T)^{(-1)^{n-i+1} \binom{n}{i}},$$

and

$$C_{f,\chi}(s, T)^{(-1)^{n-1}} = \prod_{j=0}^{+\infty} L_{f,\chi}(q^j s, T)^{\binom{n+j-1}{j}}.$$

So we have

$$C_{f,\chi}(s, T) \in 1 + s\mathbb{Z}_q[[T]][[s]].$$

We view $C_{f,\chi}(s, T)$ as a power series in the single variable s with coefficients in the T -adic complete field $\mathbb{Q}_q((T))$. The C -function $C_{f,\chi}(s, T)$ was shown T -adic entire in s by Liu [Liu2].

Let $C(\Delta)$ be the cone generated by Δ , $M(\Delta) = C(\Delta) \cap \mathbb{Z}^n$, and \deg_{Δ} the degree function on $C(\Delta)$, which is $\mathbb{R}_{\geq 0}$ linear and takes the values 1 on each co-dimension 1 face not containing 0. Let $u \in \mathbb{Z}^n/(q-1)$, and

$$M_u(\Delta) := \frac{1}{q-1} (M(\Delta) \cap u).$$

Definition 1.2. Let b be the least positive integer such that $p^b u = u$. Order elements of $\cup_{i=0}^{b-1} M_{p^i u}(\Delta)$ such that

$$\deg_{\Delta}(x_1) \leq \deg_{\Delta}(x_2) \leq \cdots.$$

A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the infinite u -twisted Hodge polygon of Δ if its slopes between consecutive integers are the numbers

$$\frac{\deg_{\Delta}(x_{bi+1}) + \deg_{\Delta}(x_{bi+2}) + \cdots + \deg_{\Delta}(x_{b(i+1)})}{b}, \quad i = 0, 1, \dots.$$

We denote it by $H_{\Delta,u}^\infty$.

The twisted Hodge polygon for Laurent polynomials can be found in the literature, see Adolphson-Sperber [AS,AS2]. Liu [Liu2] proved the following.

Theorem 1.3. *We have*

$$T - \text{adic NP of } C_{f,\chi}(s, T) \geq \text{ord}_p(q)(p-1)H_{\Delta,u}^\infty,$$

where NP is the short for Newton polygon.

If Δ is dimension one, the Newton polygon of the L -function $L_{f,\chi}(s, \pi_m)$ for $m = 1$ was studied by Blache-Férard-Zhu [BFZ], and if χ is trivial, it was studied by Liu-Liu-Niu [LLN] for $m \geq 1$.

From now on, we assume that $\Delta = [0, d]$, $\chi = \omega^{-u}$ with $1 \leq u \leq q-1$, and $q = p^b$. Write

$$u = u_0 + u_1p + \cdots + u_{b-1}p^{b-1}, 0 \leq u_i \leq p-1.$$

Definition 1.4. *For $a \in \mathbb{N}$, $1 \leq i \leq b$,*

$$\delta_{\in}^{(i)}(n) = \begin{cases} 1, & pl \equiv n - u_{b-i}(d) \text{ for some } l < d\{\frac{n}{d}\}; \\ 0, & \text{otherwise.} \end{cases}$$

where $\{\cdot\}$ is the fractional part of a real number.

Definition 1.5. *A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the twisted arithmetic polygon of $\Delta = [0, d]$ if its slopes between consecutive integers are the numbers*

$$\omega_{\Delta,u}(n) = \frac{1}{b} \sum_{i=1}^b (\lceil \frac{(p-1)n + u_{b-i}}{d} \rceil - \delta_{\in}^{(i)}(n)), a \in \mathbb{N},$$

where $\lceil \cdot \rceil$ is the least integer equal or greater than a real number. We denote it by $p_{\Delta,u}$.

Liu-Niu [LN] proved the following.

Theorem 1.6. *If $p > 4d$,*

$$T - \text{adic NP of } C_{f,\chi}(s, T) \geq bp_{\Delta,u}.$$

By a result of Li [Li], $L_{f,\chi}(s, \pi_m)$ is a polynomial with degree $p^{m-1}d$ if $p \nmid d$. Combined this result with the above theorem, one can infer the following.

Theorem 1.7. *If $p > 4d$, then*

$$\pi_m - \text{adic NP of } L_{f,\chi}(s, \pi_m) \geq bp_{\Delta,u} \text{ on } [0, p^{m-1}d],$$

with equality holding for a generic f of degree d .

We assume that the second highest exponent of f is k . So $k \leq d-1$, and

$$f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^k (a_ix^i, 0, 0, \cdots) \in W(\mathbb{F}_q[x]) \text{ with } a_da_k \neq 0.$$

For $a \in \mathbb{N}$, define

$$\begin{aligned} \varpi_{d,[0,k],u}(a) &= \frac{1}{b} \sum_{i=1}^b \left(\left\lfloor \frac{pa + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor \right) \\ &\quad + \frac{1}{b} \sum_{i=1}^b \sum_{j=0}^{r_a} \left(1_{\{\frac{r_{j,i}}{k}\} > \{\frac{r_a}{k}\}} - 1_{\{\frac{r_j}{k}\} > \{\frac{r_a}{k}\}} \right) \\ &\quad - \frac{1}{b} \sum_{i=1}^b \sum_{j=0}^{r_{a-1}} \left(1_{\{\frac{r_{j,i}}{k}\} > \{\frac{r_{a-1}}{k}\}} - 1_{\{\frac{r_j}{k}\} > \{\frac{r_{a-1}}{k}\}} \right), \end{aligned}$$

where $r_a = d\{\frac{a}{d}\}$, $r_{a,i} = d\{\frac{pa+u_{b-i}}{d}\}$ for $a \in \mathbb{N}$, $1 \leq i \leq b$.

Definition 1.8. A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the twisted arithmetic polygon of $\{d\} \cup [0, k]$ if its slopes between consecutive integers are the numbers $\varpi_{d,[0,k],u}(a)$, $a \in \mathbb{N}$. We denote it by $p_{d,[0,k],u}$.

We can prove the following.

Theorem 1.9. We have $p_{d,[0,k],u} \geq p_{\Delta,u}$.

The main result of this paper is the following.

Theorem 1.10. If $p > d(2d+1)$, then

$$T\text{-adic NP of } C_f(s, T) \geq \text{ord}_p(q) p_{d,[0,k],u}.$$

Corollary 1.11. If $p > d(2d+1)$, then

$$\pi_m\text{-adic NP of } L_{f,\chi}(s, \pi_m) \geq \text{ord}_p(q) p_{d,[0,k],u} \text{ on } [0, p^{m-1}d].$$

2. THE T -ADIC DWORK THEORY

In this section we review the T -adic analogue of Dwork theory on exponential sums.

We can write

$$\frac{u}{q-1} = -(u_0 + u_1p + \cdots), \quad u_i = u_{b+i} \text{ for } i \geq 0.$$

and $p^i u = q_i(q-1) + s_i$ for $i \in \mathbb{N}$ with $0 \leq s_i < q-1$, then $s_{b-l} = u_l + u_{l+1}p + \cdots + u_{b+l-1}p^{b-l}$ for $0 \leq l \leq b-1$ and $s_i = s_{b+i}$.

Write $C_u = \{v \in \mathbb{N} | v \equiv u \pmod{q-1}\}$. Let

$$B_u = \left\{ \sum_{v \in C_u} b_v \pi^{\frac{v}{d(q-1)}} x^{\frac{v}{q-1}} : b_v \in \mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]] \text{ and } \text{ord}_\pi b_v \rightarrow \infty \text{ as } v \rightarrow \infty \right\}.$$

Define $B = \bigoplus_{i=1}^b B_{p^i u}$, then $B = \bigoplus_{i=1}^b B_{p^i u}$ has a basis represented by

$$\prod_{1 \leq i \leq b} \{x^{\frac{s_i}{q-1} + j}\}_{j \in \mathbb{N}}.$$

Note that the Galois group of \mathbb{Q}_q over \mathbb{Q}_p can act on B but keeping $\pi^{1/d}$ as well as the variable x fixed. Let $\sigma \in \text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ be the Frobenius element such that $\sigma(\zeta) = \zeta^p$ if ζ is a $(q-1)$ -th root of unity and Ψ_p the operator on B defined by the formula

$$\Psi_p\left(\sum_{i \in \mathbb{N}} c_i x^i\right) = \sum_{i \in \mathbb{N}} c_{pi} x^i.$$

The Frobenius operator Ψ on B is defined by $\Psi := \sigma^{-1} \circ \Psi_p \circ E_f$, where $E_f(x) := E(\pi \hat{a}_d x^d) \prod_{i=1}^k E(\pi \hat{a}_i x^i)$.

Note that $\Psi : B_u \rightarrow B_{p^{-1}u} = B_{p^{b-1}u}$, hence Ψ is well defined. It follows that Ψ^b operates on B_u and is linear over $\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]$. Moreover it is completely continuous in the sense of Serre [Se].

Theorem 2.1 (T -adic Dwork trace formula).

$$C_{f,\chi}(s, T) = \det(1 - \Psi^b s | B_u / \mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]).$$

3. KEY ESTIMATE

In order to study

$$C_{f,\chi}(s, T) = \det(1 - \Psi^b s | B_u / \mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]),$$

we first study

$$\det(1 - \Psi s | B / \mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]) = \sum_{i=0}^{\infty} (-1)^i c_i s^i.$$

We are going to show that

Theorem 3.1. *If $p > d(2d+1)$, then we have*

$$\text{ord}_\pi(c_{b^2m}) \geq b^2 p_{d,[0,k],u}(m).$$

Consider the operator $\Psi_p \circ E_f(x)$ on B , we have

$$\begin{aligned} \Psi_p \circ E_f(x)(x^{\frac{s_i}{q-1}+j}) &= \Psi_p\left(\sum_{l=0}^{\infty} \gamma_l x^{\frac{s_i}{q-1}+j+l}\right) \\ &= \sum_{pl+s_i-j \geq 0} \gamma_{pl+s_i-j} x^{l+\frac{p^{b-1}s_i}{q-1}} \\ &= \sum_{l=0}^{\infty} \gamma_{pl+u_{b-i}-j} x^{l+\frac{s_i-1}{q-1}}. \end{aligned}$$

Then the matrix of $\Psi_p \circ E_f(x)$ on B with respect to the basis $\prod_{1 \leq i \leq b} \{x^{\frac{s_i}{q-1}+j}\}_{j \in \mathbb{N}}$ is

$$(G_{(k,l)}(i,j))_{1 \leq k, i \leq b, l, j \in \mathbb{N}}.$$

where

$$G_{(k,l)(i,j)} = \begin{cases} \gamma_{pl+u_{b-i}-j}, & k = i - 1; \\ 0, & \text{otherwise.} \end{cases}$$

Fix a normal basis $\bar{\xi}_1, \dots, \bar{\xi}_b$ of \mathbb{F}_q over \mathbb{F}_p . Let ξ_1, \dots, ξ_b be their Teichmüller lifts. Then ξ_1, \dots, ξ_b is a normal basis of \mathbb{Q}_q over \mathbb{Q}_p , and σ acts on ξ_1, \dots, ξ_b as a permutation. Write

$$\xi_v^{\sigma^{-1}} G_{(k,l)(i,j)}^{\sigma^{-1}} = \sum_{w=1}^b G_{((k,l),w)((i,j),v)} \xi_w.$$

It is easy to see that $G_{((k,l),w)((i,j),v)} = 0$ if $k \neq i - 1$. For $k = i - 1$, write

$$G_{((i-1,l),w)((i,j),v)} = G_{(l,w)(j,v)}^{(i)}.$$

Write $G^{(i)} = (G_{(l,w)(j,v)}^{(i)})_{l,j \in \mathbb{N}, 1 \leq w, v \leq b}$, then the matrix of the operator Ψ on B over $\mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]$ with respect to the basis $\{\xi_v x^{\frac{s_i}{q-1} + j}\}_{1 \leq i, v \leq b; j \in \mathbb{N}}$ is

$$G = \begin{pmatrix} 0 & G^{(1)} & 0 & \dots & 0 \\ 0 & 0 & G^{(2)} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & G^{(b-1)} \\ G^{(b)} & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Hence we have

$$\det(1 - \Psi s | B / \mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]) = \det(1 - Gs) = \sum_{m=0}^{\infty} (-1)^m c_m s^m,$$

with $c_m = \sum_F \det(F)$, where F runs over all principle $m \times m$ submatrix of G .

For every principle submatrix F of G , write $F^{(i)} = F \cap G^{(i)}$ as the submatrix of $G^{(i)}$. For principle $bm \times bm$ submatrix F of G , by linear algebra, if one of $F^{(i)}$ is not $m \times m$ submatrix of $G^{(i)}$, then at least one row or column of F are 0 since F is principle.

Let \mathcal{F}_m be the set of all $bm \times bm$ principle submatrices F of G with $F^{(i)}$ all $m \times m$ submatrices of $G^{(i)}$ for each $1 \leq i \leq b$.

Lemma 3.2. *We have*

$$c_{bm} = \sum_{F \in \mathcal{F}_m} \det(F) = \sum_{F \in \mathcal{F}_m} (-1)^{m^2(b-1)} \prod_{i=1}^b \det(F^{(i)}).$$

Corollary 3.3.

$$c_{b^2m} = \sum_{F \in \mathcal{F}_{bm}} \prod_{i=1}^b \det(F^{(i)}) = \sum_{F \in \mathcal{F}_{bm}} \prod_{i=1}^b \left(\sum_{\tau} \text{sgn}(\tau) \prod_{(l,w) \in R_i} G_{(l,w)\tau(l,w)}^{(i)} \right),$$

where R_i runs over all subsets of $\mathbb{N} \times \{1, 2, \dots, b\}$ with cardinality bm , τ runs over all permutations of R_i , $1 \leq i \leq b$.

So Theorem 3.1 is reduced to the following.

Theorem 3.4. *Let $p > d(2d + 1)$. Then we have*

$$\sum_{i=1}^b \text{ord}_\pi \left(\sum_{\tau} \text{sgn}(\tau) \prod_{(l, \omega) \in R_i} G_{(l, \omega)\tau(l, \omega)}^{(i)} \right) \geq b^2 p_{d, [0, k], u}(m).$$

Let $O(\pi^\alpha)$ denote any element of π -adic order $\geq \alpha$.

Lemma 3.5. *For any $1 \leq i \leq b$ and $1 \leq w, v \leq b$, we have*

$$G_{(l, w)(j, v)}^{(i)} = O(\pi^{\lfloor \frac{pl + u_{b-i-j}}{d} \rfloor + \lceil \frac{d \lfloor \frac{pl + u_{b-i-j}}{d} \rfloor}{k} \rceil}).$$

Proof. Write

$$E_f(x) = \sum_{i \in \mathbb{N}} \gamma_i x^i.$$

Then

$$\gamma_i = \sum_{\substack{dn_d + \sum_{j=1}^k jn_j = i \\ n_j \geq 0}} \pi^{\sum_{j=1}^k n_j + n_d} \prod_{j=1}^k \lambda_{n_j} \hat{a}_j^{n_j} \lambda_{n_d} \hat{a}_d^{n_d} = O(\pi^{\lfloor \frac{i}{d} \rfloor + \lceil \frac{r_i}{k} \rceil}).$$

Since

$$\xi_v^{\sigma^{-1}} G_{(i-1, l)(i, j)}^{\sigma^{-1}} = \xi_v^{\sigma^{-1}} \gamma_{pl + u_{b-i-j}}^{\sigma^{-1}} = \sum_{w=1}^b G_{(l, w)(j, v)}^{(i)} \xi_w,$$

we have

$$\text{ord}_\pi(G_{(l, w)(j, v)}^{(i)}) = \text{ord}_\pi(\gamma_{pl + u_{b-i-j}}).$$

The lemma now follows. \square

By the above lemma, Theorem 3.4 is reduced to the following.

Theorem 3.6. *Let $p > d(2d + 1)$. For $1 \leq i \leq b$, let $R_i \subset \mathbb{N} \times \{1, 2, \dots, b\}$ be a subset of cardinality bm , and τ a permutation of R_i . Then*

$$\sum_{i=1}^b \sum_{(l, \omega) \in R_i} (\lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \rfloor + \lceil \frac{d \lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \rfloor}{k} \rceil) \geq b^2 p_{d, [0, k], u}(m),$$

where $\tau(l)$ is defined by $\tau(l, \omega) = (\tau(l), \tau(\omega))$.

Proof. By definition, we have

$$p_{d, [0, k], u}(m) = \sum_{a=0}^{m-1} \varpi_{d, [0, k], u}(a) = \sum_{i=1}^b p_{d, [0, k], u}^{(i)}(m),$$

where

$$p_{d,[0,k],u}^{(i)}(m) = \frac{1}{b} \sum_{a=0}^{m-1} \left(\left\lfloor \frac{pa + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{r_{a,i}}{k} \right\rfloor - \left\lfloor \frac{r_a}{k} \right\rfloor + 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_{m-1}}{k}\}} - 1_{\{\frac{r_a}{k}\} > \{\frac{r_{m-1}}{k}\}} \right).$$

Then it suffices to show that for $1 \leq i \leq b$, and for any permutation τ of R_i , we have

$$\sum_{(l,\omega) \in R_i} \left(\left\lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \right\rfloor + \left\lceil \frac{d \left\lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \right\rfloor}{k} \right\rceil \right) \geq b^2 p_{d,[0,k],u}^{(i)}(m).$$

Note that

$$\begin{aligned} & \sum_{(l,\omega) \in R_i} \left(\left\lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \right\rfloor + \left\lceil \frac{d \left\lfloor \frac{pl + u_{b-i} - \tau(l)}{d} \right\rfloor}{k} \right\rceil \right) \\ &= \sum_{(l,\omega) \in R_i} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \left\{ \frac{pl + u_{b-i}}{d} \right\} - \left\{ \frac{\tau(l)}{d} \right\} \right\rfloor + \left\lceil \frac{r_{l,i} - r_{\tau(l)} - d \left[\left\{ \frac{pl}{d} \right\} - \left\{ \frac{\tau(l)}{d} \right\} \right]}{k} \right\rceil \right) \\ &= \sum_{(l,\omega) \in R_i} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor + \left\lceil \frac{(d-k)1_{r_{\tau(l)} > r_{l,i}} + \{ \frac{r_{l,i}}{k} \} - \{ \frac{r_{\tau(l)}}{k} \}}{k} \right\rceil \right). \end{aligned}$$

And

$$\begin{aligned} & \sum_{(l,\omega) \in R_i} \left(\left\lceil \frac{(d-k)1_{r_{\tau(l)} > r_{l,i}} + \{ \frac{r_{l,i}}{k} \} - \{ \frac{r_{\tau(l)}}{k} \}}{k} \right\rceil \right) \\ & \geq \sum_{(l,\omega) \in R_i} 1_{\left\{ \frac{r_{\tau(l)}}{k} \right\} \leq \left\{ \frac{r_{m-1}}{k} \right\} < \left\{ \frac{r_{l,i}}{k} \right\}} \\ & \geq \sum_{(l,\omega) \in R_i} 1_{\left\{ \frac{r_{l,i}}{k} \right\} > \left\{ \frac{r_{m-1}}{k} \right\}} - \sum_{(l,\omega) \in R_i} 1_{\left\{ \frac{r_l}{k} \right\} > \left\{ \frac{r_{m-1}}{k} \right\}}. \end{aligned}$$

We have

$$\begin{aligned} & \sum_{(l,\omega) \in R_i} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor \right) \\ &= b \sum_{l=0}^{m-1} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor \right) + \sum_{\substack{(l,\omega) \in R_i \\ l \geq m}} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor \right) \\ & \quad - \sum_{\substack{(l,\omega) \notin R_i \\ 0 \leq l < m}} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor \right) \\ & \geq b \sum_{l=0}^{m-1} \left(\left\lfloor \frac{pl + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{l}{d} \right\rfloor + \left\lfloor \frac{r_{l,i}}{k} \right\rfloor - \left\lfloor \frac{r_l}{k} \right\rfloor \right) + N \left(\left\lfloor \frac{pm + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{m}{d} \right\rfloor - \left\lfloor \frac{d-1}{k} \right\rfloor \right) \\ & \quad - N \left(\left\lfloor \frac{p(m-1) + u_{b-i}}{d} \right\rfloor - \left\lfloor \frac{m-1}{d} \right\rfloor + \left\lfloor \frac{d-1}{k} \right\rfloor \right) \end{aligned}$$

$$\geq b \sum_{l=0}^{m-1} ([\frac{pl + u_{b-i}}{d}] - [\frac{l}{d}] + [\frac{r_{l,i}}{k}] - [\frac{r_l}{k}]) + N([\frac{p}{d}] - 1 - 2(d-1)),$$

where $N = \#\{(l, \omega) \in R_i | l \geq m\} = \#\{(l, \omega) \notin R_i | 0 \leq l < m\}$.

Similarly, we have

$$\sum_{(l, \omega) \in R_i} 1_{\{\frac{r_{l,i}}{k} > \{\frac{r_{m-1}}{k}\}\}} \geq b \sum_{l=0}^{m-1} 1_{\{\frac{r_{l,i}}{k} > \{\frac{r_{m-1}}{k}\}\}} - N,$$

and

$$\sum_{(l, \omega) \in R} 1_{\{\frac{r_l}{k} > \{\frac{r_{m-1}}{k}\}\}} \leq b \sum_{l=0}^{m-1} 1_{\{\frac{r_l}{k} > \{\frac{r_{m-1}}{k}\}\}} + N.$$

Therefore,

$$\begin{aligned} & \sum_{(l, \omega) \in R_i} ([\frac{pl + u_{b-i} - \tau(l)}{d}] + [\frac{d\{\frac{pl + u_{b-i} - \tau(l)}{d}\}}{k}]) \\ & \geq b \sum_{l=0}^{m-1} ([\frac{pl + u_{b-i}}{d}] - [\frac{l}{d}] + [\frac{r_{l,i}}{k}] - [\frac{r_l}{k}] + 1_{\{\frac{r_{l,i}}{k} > \{\frac{r_{m-1}}{k}\}\}} - 1_{\{\frac{r_l}{k} > \{\frac{r_{m-1}}{k}\}\}}) \\ & \quad + N([\frac{p}{d}] - 1 - 2(d-1) - 2) \\ & = b^2 p_{d, [0, k], u}^{(i)}(m) + N([\frac{p}{d}] - 2d + 1) \geq b^2 p_{d, [0, k], u}^{(i)}(m). \end{aligned}$$

□

4. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.10, which says that, if $p > d(2d+1)$, then

$$T - \text{adic NP of } C_{f, \chi}(s, T) \geq b p_{d, [0, k], u}.$$

Lemma 4.1. *We have*

$$\begin{aligned} & T - \text{adic NP of } \det(1 - \Psi^b s^b | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]) \\ & = T - \text{adic NP of } \det(1 - \Psi s | B/\mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]) . \end{aligned}$$

Proof. The lemma follows from the following:

$$\begin{aligned} \prod_{\zeta^b=1} \det(1 - \Psi \zeta s | B/\mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]) &= \det(1 - \Psi^b s^b | B/\mathbb{Z}_p[[\pi^{\frac{1}{d(q-1)}}]]) \\ &= \text{Norm}(\det(1 - \Psi^b s^b | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]) , \end{aligned}$$

where the Norm is the norm map from $\mathbb{Q}_q[[\pi^{\frac{1}{d(q-1)}}]]$ to $\mathbb{Q}_p[[\pi^{\frac{1}{d(q-1)}}]]$. □

Lemma 4.2. *We have*

$$T - \text{adic NP of } C_{f, \chi}(s, T)^b = T - \text{adic NP of } \det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]) .$$

Proof. Let σ act on $\mathbb{Q}_q[[T]]$ coordinate-wise. Hence

$$\begin{aligned} S_{f,\chi^p}(l, T) &= \sum_{x \in \mathbb{F}_{q^l}^\times} \chi(\text{Norm}_{\mathbb{F}_{q^l}/\mathbb{F}_q}(x))^p (1+T)^{\text{Tr}_{\mathbb{Q}_{q^l}/\mathbb{Q}_p}(f(x))} \\ &= S_{f,\chi}(l, T)^\sigma, \end{aligned}$$

therefore $C_{f,\chi^p}(s, T) = C_{f,\chi}(s, T)^\sigma$, which yields that the T-adic Newton polygons of $C_{f,\chi^p}(s, T)$ and $C_{f,\chi}(s, T)$ coincide with each other. Hence the lemma follows from the following

$$\begin{aligned} \prod_{i=1}^b C_{f,\chi}(s, T)^{\sigma^i} &= \prod_{i=1}^b C_{f,\chi^{p^i}}(s, T) \\ &= \prod_{i=1}^b \det(1 - \Psi^b s | B_{p^{i_u}}/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]) \\ &= \det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]) . \end{aligned}$$

□

Corollary 4.3. *The T-adic Newton polygon of $C_{f,\chi}(s, T)$ is the lower convex closure of the points*

$$(i, \frac{1}{b} \text{ord}_\pi c_{b^2 i}), \quad i = 0, 1, \dots$$

Proof. By Lemma 4.1, the T-adic Newton polygon of

$$\det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]])$$

is the lower convex closure of the points

$$(bi, \text{ord}_\pi c_{bi}), \quad i = 0, 1, \dots$$

Hence the T-adic Newton polygon of

$$\det(1 - \Psi^b s | B/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]])$$

is the convex closure of the points

$$(i, \text{ord}_\pi c_{bi}), \quad i = 0, 1, \dots$$

By Lemma 4.2, the T-adic Newton polygon of $C_{f,\chi}(s, T)^b$ is the lower convex closure of the points

$$(bi, \text{ord}_\pi c_{b^2 i}), \quad i = 0, 1, \dots$$

The lemma is proved. □

We now prove Theorem 1.10.

Proof of Theorem 1.10. By Theorem 2.1, we have

$$C_{f,\chi}(s, T) = \det(1 - \Psi^b s | B_u/\mathbb{Z}_q[[\pi^{\frac{1}{d(q-1)}}]]),$$

Then by Corollary 4.3, the T -adic Newton polygon of $C_{f,\chi}(s, T)$ is the lower convex closure of the points

$$(i, \frac{1}{b} \text{ord}_\pi c_{b^2 i}), \quad i = 0, 1, \dots.$$

Therefore the result follows from Theorem 3.1, which says that, if $p > d(2d + 1)$, then we have

$$\text{ord}_\pi(c_{b^2 m}) \geq b^2 p_{d,[0,k],u}(m).$$

□

We conclude this section by proving Corollary 1.11.

Proof of Corollary 1.11. Assume that $L_{f,\chi}(s, \pi_m) = \prod_{i=1}^{p^{m-1}d} (1 - \beta_i s)$. Then

$$C_{f,\chi}(s, \pi_m) = \prod_{j=0}^{\infty} L_{f,\chi}(q^j s, \pi_m) = \prod_{j=0}^{\infty} \prod_{i=1}^{p^{m-1}d} (1 - \beta_i q^j s).$$

Therefore the slopes of the q -adic Newton polygon of $C_{f,\chi}(s, \pi_m)$ are the numbers

$$j + \text{ord}_q(\beta_i), \quad 1 \leq i \leq p^{m-1}d, j = 0, 1, \dots.$$

It is well-known that $\text{ord}_q(\beta_i) \leq 1$ for all i . Therefore,

$$q\text{-adic NP of } L_{f,\chi}(s, \pi_m) = q\text{-adic NP of } C_{f,\chi}(s, \pi_m) \text{ on } [0, p^{m-1}d].$$

It follows that

$$\pi_m\text{-adic NP of } L_{f,\chi}(s, \pi_m) = \pi_m\text{-adic NP of } C_{f,\chi}(s, \pi_m) \text{ on } [0, p^{m-1}d].$$

By the integrality of $C_{f,\chi}(s, T)$ and Theorem 1.10, we have

$$\pi_m\text{-adic NP of } C_{f,\chi}(s, \pi_m) \geq T\text{-adic NP of } C_{f,\chi}(s, T) \geq \text{ord}_p(q) p_{d,[0,k],u}.$$

Therefore,

$$\pi_m\text{-adic NP of } L_{f,\chi}(s, \pi_m) \geq \text{ord}_p(q) p_{d,[0,k],u} \text{ on } [0, p^{m-1}d].$$

□

5. COMPARISON BETWEEN ARITHMETIC POLYGONS

In this section we prove Theorem 1.9, which says that

$$p_{d,[0,k],u} \geq p_{\Delta,u}.$$

Proof of Theorem 1.9 It is clear that $p_{d,[0,k]}(0) = p_{\Delta}(0)$. It suffices to show that, for $m \in \mathbb{N}$, we have

$$p_{d,[0,k],u}(m+1) \geq p_{\Delta,u}(m+1).$$

By a result in Liu-Niu [LN], we have

$$p_{\Delta,u}(m+1) = \sum_{i=1}^b p_{\Delta,u}^{(i)}(m+1),$$

where

$$p_{\Delta,u}^{(i)}(m+1) = \frac{1}{b} \left(\sum_{a=0}^m \left(\left\lceil \frac{pa + u_{b-i}}{d} \right\rceil - \left\lceil \frac{a}{d} \right\rceil \right) + \sum_{a=0}^{r_m} \left(1_{\{\frac{a}{d}\}' \leq \frac{r_m}{d} < \{\frac{pa+u_{b-i}}{d}\}'} - 1_{\{\frac{u_{b-i}}{d}\}' \leq \frac{r_m}{d}} \right) \right).$$

By definition, we have

$$p_{d,[0,k],u}(m+1) = \sum_{a=0}^m \varpi_{d,[0,k],u}(a) = \sum_{i=1}^b p_{d,[0,k],u}^{(i)}(m+1),$$

where

$$p_{d,[0,k],u}^{(i)}(m+1) = \frac{1}{b} \sum_{a=0}^m \left(\left\lceil \frac{pa + u_{b-i}}{d} \right\rceil - \left\lceil \frac{a}{d} \right\rceil + \left\lceil \frac{r_{a,i}}{k} \right\rceil - \left\lceil \frac{r_a}{k} \right\rceil + 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_m}{k}\}} - 1_{\{\frac{r_a}{k}\} > \{\frac{r_m}{k}\}} \right).$$

Then it suffices to show that

$$p_{d,[0,k],u}^{(i)}(m+1) \geq p_{\Delta,u}^{(i)}(m+1).$$

For $m \geq 0$, $1 \leq i \leq b$, let $A_{i1} = \{0 \leq a \leq r_m | a \neq r_{l,i} \text{ for some } 0 \leq l \leq r_m\}$.

Note that

$$\{a : 0 \leq a \leq r_m\} = A_{i1} \cup A_{i2},$$

where $A_{i2} = \{r_{a,i} | 0 \leq a, r_{a,i} \leq r_m\}$. And

$$\{r_{a,i} | 0 \leq a \leq r_m\} = A_{i2} \cup A_{i3},$$

where $A_{i3} = \{r_{a,i} > r_m | 0 \leq a \leq r_m\}$, so we have $|A_{i1}| = |A_{i3}|$. Then

$$\begin{aligned} & \sum_{a=0}^m \left(\left\lceil \frac{r_{a,i}}{k} \right\rceil - \left\lceil \frac{r_a}{k} \right\rceil + 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_m}{k}\}} - 1_{\{\frac{r_a}{k}\} > \{\frac{r_m}{k}\}} \right) \\ &= \sum_{a=0}^{r_m} \left(\left\lceil \frac{r_{a,i}}{k} \right\rceil - \left\lceil \frac{a}{k} \right\rceil + 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_m}{k}\}} - 1_{\{\frac{a}{k}\} > \{\frac{r_m}{k}\}} \right) \\ &= \sum_{r_{a,i} \in A_{i3}} \left\lceil \frac{r_{a,i}}{k} \right\rceil - \sum_{a \in A_{i1}} \left\lceil \frac{a}{k} \right\rceil + \sum_{r_{a,i} \in A_{i3}} 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_m}{k}\}} - \sum_{a \in A_{i1}} 1_{\{\frac{a}{k}\} > \{\frac{r_m}{k}\}} \\ &= \sum_{r_{a,i} \in A_{i3}} \left\lceil \frac{r_{a,i} - r_m}{k} \right\rceil + \sum_{a \in A_{i1}} \left(\left\lceil \frac{r_m - a}{k} \right\rceil - 1_{\{\frac{a}{k}\} > \{\frac{r_m}{k}\}} - 1_{\{\frac{a}{k}\} < \{\frac{r_m}{k}\}} \right) \\ &\geq \sum_{r_{a,i} \in A_{i3}} \left\lceil \frac{r_{a,i} - r_m}{k} \right\rceil. \end{aligned}$$

Therefore, we have

$$\begin{aligned} & b(p_{d,[0,k],u}^{(i)}(m+1) - p_{\Delta,u}^{(i)}(m+1)) \\ &= \sum_{a=0}^m \left(\left\lceil \frac{pa + u_{b-i}}{d} \right\rceil - \left\lceil \frac{a}{d} \right\rceil + \left\lceil \frac{r_{a,i}}{k} \right\rceil - \left\lceil \frac{r_a}{k} \right\rceil + 1_{\{\frac{r_{a,i}}{k}\} > \{\frac{r_m}{k}\}} - 1_{\{\frac{r_a}{k}\} > \{\frac{r_m}{k}\}} \right) \\ &- \left(\sum_{a=0}^m \left(\left\lceil \frac{pa + u_{b-i}}{d} \right\rceil - \left\lceil \frac{a}{d} \right\rceil \right) + \sum_{a=0}^{r_m} \left(1_{\{\frac{a}{d}\}' \leq \frac{r_m}{d} < \{\frac{pa+u_{b-i}}{d}\}'} - 1_{\{\frac{u_{b-i}}{d}\}' \leq \frac{r_m}{d}} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &\geq \sum_{a=0}^m (\lceil \frac{a}{d} \rceil - \lceil \frac{r_{a,i}}{d} \rceil) + \sum_{r_{a,i} \in A_{i3}} \lceil \frac{r_{a,i} - r_m}{k} \rceil - \sum_{a=1}^{r_m} 1_{\frac{r_m}{d} < \{\frac{pa+u_{b-i}}{d}\}'} + 1_{\{\frac{u_{b-i}}{d}\}' \leq \frac{r_m}{d}} \\
 &= \sum_{a=0}^{r_m} (\lceil \frac{a}{d} \rceil - \lceil \frac{r_{a,i}}{d} \rceil) + \sum_{a=1}^{r_m} (\lceil \frac{r_{a,i} - r_m}{k} \rceil - 1) 1_{r_{a,i} > r_m} + \lceil \frac{r_{0,i} - r_m}{k} \rceil 1_{\{\frac{u_{b-i}}{d}\}' > \frac{r_m}{d}} \\
 &\quad + \delta_{m,2}^{(i)} + 1_{\{\frac{u_{b-i}}{d}\}' \leq \frac{r_m}{d}} \\
 &\geq \delta_{m,1}^{(i)} + \delta_{m,2}^{(i)} + 1_{\{\frac{u_{b-i}}{d}\}' > \frac{r_m}{d}} + 1_{\{\frac{u_{b-i}}{d}\}' \leq \frac{r_m}{d}} \geq 0.
 \end{aligned}$$

where for $1 \leq i \leq b$,

$$\delta_{m,1}^{(i)} = \begin{cases} 0, & \text{if there exists } 0 \leq a \leq r_m \text{ such that } pa + u_{b-i} \equiv 0 \pmod{d}; \\ -1, & \text{otherwise.} \end{cases}$$

$$\delta_{m,2}^{(i)} = \begin{cases} -1, & \text{if there exists } 1 \leq a \leq r_m \text{ such that } pa + u_{b-i} \equiv 0 \pmod{d}; \\ 0, & \text{otherwise.} \end{cases}$$

The theorem now follows. \square

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